REDUCED ORDER MODELING OF MISTUNED BLADED DISKS CONSIDERING AERODYNAMIC COUPLING AND MODE FAMILY INTERACTION

S. Willeke$^1$ - C. Keller$^2$ - L. Panning-von Scheidt$^1$ - J. Seume$^2$ - J. Wallaschek$^1$

Leibniz Universität Hannover

$^1$Institute of Dynamics and Vibration Research (IDS), Appelstraße 11
$^2$Institute of Turbomachinery and Fluid Dynamics (TFD), Appelstraße 9
30167 Hannover, Germany
willeke@ids.uni-hanover.de

ABSTRACT
A substructure-based reduced order model for the numerical prediction of the mistuned dynamics of bladed disks is presented. The structural mistuning is introduced to the tuned disk by blade-to-blade variations of the natural frequencies. Aeroelastic influence coefficients provide aerodynamic inter-blade and inter-modal coupling via the fluid flow. The accuracy and efficiency of the reduced modeling approach are highlighted by a comparison with conventional FEA simulations and unsteady CFD results. In total, the model reduction provides a computational saving of up to 90% while predicting the amplitudes of forced vibrations within a tolerance of 0.7%. The proposed modeling technique is used to analyze the forced response and the aeroelastic stability of an axial compressor blisk. This exemplary study reveals an attenuation of the mistuned response due to an increase in aerodynamic damping. The intentionally provoked interaction of two mode families illustrates the significance of the inter-modal aerodynamic coupling.

KEYWORDS
Aeroelasticity, Forced Response, Flutter Stability, Compressor Blisk

NOMENCLATURE

<table>
<thead>
<tr>
<th>CMS</th>
<th>Component Mode Synthesis</th>
<th>SMT</th>
<th>Secondary Modal Truncation</th>
</tr>
</thead>
<tbody>
<tr>
<td>EO</td>
<td>Engine Order</td>
<td>WBS</td>
<td>Wave-Based Substructuring</td>
</tr>
<tr>
<td>FEA</td>
<td>Finite Element Analysis</td>
<td>$k_{aero}$</td>
<td>aerodynamic coupling</td>
</tr>
<tr>
<td>IBPA</td>
<td>Inter-Blade Phase Angle</td>
<td>$k_{aero}^{(i,j)}$</td>
<td>intra-modal coupling terms</td>
</tr>
<tr>
<td>MAC</td>
<td>Modal Assurance Criterion</td>
<td>$k_{aero}^{(i,j)}$</td>
<td>inter-modal coupling terms</td>
</tr>
<tr>
<td>MPC</td>
<td>Multi-Point Constraints</td>
<td>$q$</td>
<td>modal displacement</td>
</tr>
<tr>
<td>ND</td>
<td>Nodal Diameter</td>
<td>$u$</td>
<td>physical displacement</td>
</tr>
<tr>
<td>ROM</td>
<td>Reduced Order Model</td>
<td>$\zeta$</td>
<td>aerodynamic damping</td>
</tr>
</tbody>
</table>

INTRODUCTION
The geometrical complexity of modern blade designs is mainly driven by the effort to increase turbomachinery efficiency. While strongly three-dimensional blade shapes provide a considerable gain in aerodynamic performance, they often increase the blades’ susceptibility to structural vibrations. Since blade vibrations are accompanied by an aggravated risk of high cycle fatigue, the accurate prediction of bladed disk dynamics plays an essential part in the turbomachinery design process.
The Finite Element Analysis (FEA) represents an established approach for the numerical simulation of bladed disk vibrations. While cyclic symmetric FE models of a single segment benefit from reduced computational costs, they occasionally fail to predict the actual dynamics of real structures. Such a discrepancy is often related to slight variations between the bladed segments stemming from manufacturing tolerances and operational wear. This mistuning breaks the cyclic symmetry of the bladed disk and yields localized modes (Castanier and Pierre, 2006). When exposed to external excitation, the mistuned structure may suffer from increased vibration amplitudes compared to the forced response of the tuned configuration. Thus, the individual properties of each component have to be considered in the numerical models.

Several authors have proposed reduced order models (ROM) for the efficient solution of mistuned problems. On the one hand, various substructuring techniques have been applied such as the fixed-interface Component Mode Synthesis (CMS) by Craig and Bampton (1968) and free-interface methods by Craig and Chang (1976). On the other hand, methods based on the overall system modes have been presented like the Subset of Nominal Modes approach (SNM) by Yang and Griffin (1999) and the Fundamental Mistuning Model (FMM) by Feiner and Griffin (2002). A recent comparison between these two ROM groups is presented by Gutierrez Salas et al. (2016). The study reveals a lack of accuracy in the forced response prediction of mistuned bladed disks by the SNM approach which is related to the assumption of cyclic symmetry. The prediction by a CMS model without this assumption shows a good agreement with FE results.

Besides the development of novel reduction techniques, ongoing research is focused on the extension of established ROM approaches by multidisciplinary aspects. An area that has attracted increasing attention is the field of aeroelasticity covering the interaction of vibrating structures and fluid flows. In particular, the prevention of self-excited blade flutter by intentional mistuning is studied by (Martel et al., 2008). It is shown that the localized vibration leads to an aeroelastic stabilization of weakly damped modes. Similar results for a mistuned compressor fan are obtained using the Multimode Least Square (MLS) by Mayorca et al. (2012). In combination with a Guyan-reduced model, the MLS approach considers the aerodynamic coupling between various mode families. Since closely spaced modes may interact in the presence of mistuning, this method supersedes approaches, which are restricted to a single mode family.

The present paper is focused on the aeroelastic extension of a structural reduced order model presented by Hohl et al. (2009). First, the reduction steps of the original approach are summarized prior to addressing its efficiency and accuracy. Next, the aerodynamic coupling matrices are derived from unsteady CFD simulations and added to the equation of motion. Finally, the extended model is used for the forced response and stability analysis of a compressor blisk. The results illustrate the significance of aerodynamic coupling for tuned and mistuned blisks.

**REDUCED ORDER MODELING OF MISTUNED BLADED DISK DYNAMICS**

Based on a spatial discretization with finite elements, the dynamic behavior of a bladed disk is governed by the following equation of motion,

\[
\left[ -\omega^2 M + i\omega \left( C_v + C_{aero} \right) + iC_s + K + K_{aero} \right] \hat{u} = \hat{f}_e .
\]

The structural mass and stiffness of the bladed disk are modeled by the sparse matrices $M$ and $K$. The aerodynamic coupling is introduced as an additional stiffness matrix $K_{aero}$ and a damping matrix $C_{aero}$. The vector $\hat{u}$ denotes the structural displacement amplitudes. Structural and proportional Rayleigh damping are added by the matrices $C_s$ and $C_v$ respectively,

\[
C_s = d_0 K \quad \text{and} \quad C_v = \alpha M + \beta K .
\]
The vector $\hat{f}_e$ comprises the amplitudes of the excitation forces $f_{e,k}$ acting with an angular frequency $\Omega$ on the blade $k$:

$$f_{e,k}(t) = \hat{f}_{e,k} e^{i\Omega t} \quad \text{where} \quad \hat{f}_{e,k} = \hat{f}_{e,1} e^{i\frac{2\pi}{N} EO(k-1)} \quad \text{for} \quad k = 1, 2, \ldots, N . \quad (3)$$

The phase shift between the amplitudes $\hat{f}_{e,1}$ and $\hat{f}_{e,k}$ of the first and $k$-th blade depends on the number of blades $N$ and the engine order (EO).

The dimension of Eq. (1) is specified by the FE discretization of the bladed disk. Since the model consists of the full disk annulus and each individual blade, the vector $\hat{u}$ may contain numerous degrees of freedom (dof). The multi-step reduction procedure by Hohl et al. (2009) is used to transform the problem to modal subspaces of gradually smaller dimension. In the following sections, the reduction steps of a Component Mode Synthesis (CMS), a Wave-Based Substructuring (WBS), and a Secondary Modal Truncation (SMT) are summarized.

**Component Mode Synthesis**

First, the bladed disk is partitioned into blade and disk components (superscripts $b$ and $d$), leading to system matrices of the form

$$Z^{(s)} = \begin{bmatrix} Z^{(s)}_{\Gamma\Gamma} & Z^{(s)}_{\Gamma\Xi} \\ Z^{(s)}_{\Xi\Gamma} & Z^{(s)}_{\Xi\Xi} \end{bmatrix} \quad \text{where} \quad Z^{(s)} = M^{(b)} , M^{(d)} , K^{(b)} , K^{(d)} . \quad (4)$$

Next, the degrees of freedom of each substructure $s$ are reduced according to the fixed-interface component mode synthesis by Craig and Bampton (1968). Herein, the displacement $u^{(s)}_{\Xi}$ of each substructure is approximated by a limited set of normal modes $\Phi^{(s)}$ with fixed boundary conditions along the blade-disk-interface $\Gamma$. The modal basis $\Phi^{(s)}$ of dynamic component modes is enriched by adding a set $\Psi^{(s)}$ of static constraint modes with a unitary displacement at each interface degree of freedom,

$$K^{(s)} \Phi^{(s)} = M^{(s)} \Phi^{(s)} \Lambda^{(s)} \quad \text{and} \quad \Psi^{(s)} = -K^{(s)}_{\Xi\Xi}^{-1} K^{(s)}_{\Xi\Xi} . \quad (5)$$

A synthesis of the mode sets $\Phi^{(s)}$ and $\Psi^{(s)}$ yields the transformation matrix $T_{\text{cms}}^{(s)}$ which is used to obtain the CMS-reduced matrix $Z_{\text{cms}}^{(s)}$.

$$u^{(s)} = \begin{pmatrix} u^{(s)}_{\Gamma} \\ u^{(s)}_{\Xi} \end{pmatrix} = \begin{bmatrix} I & 0 \\ \Psi^{(s)} & \Phi^{(s)} \end{bmatrix} \begin{pmatrix} u^{(s)}_{\Gamma} \\ u^{(s)}_{\Xi} \end{pmatrix} \quad \text{and} \quad Z_{\text{cms}}^{(s)} = T_{\text{cms}}^{(s) H} Z^{(s)} T_{\text{cms}}^{(s)} . \quad (6)$$

The CMS-transformation of the matrix $Z^{(s)}$ leads to a reduced matrix $Z_{\text{cms}}^{(s)}$

$$Z_{\text{cms}}^{(s)} = T_{\text{cms}}^{(s) H} Z^{(s)} T_{\text{cms}}^{(s)} \quad (7)$$

for each substructure $s$. It should be noted that the CMS-reduced matrices of the disk are transformed to cyclic coordinates. In order to reassemble each blade with the tuned disk, the degrees of freedom along the interface $\Gamma$ are expressed in non-cyclic coordinates. This allows a mode-specific frequency mistuning of every blade individually. A non-conforming discretization of the interface between the disk and blades is taken into account by imposing linear Multi-Point Constraints (MPC) along the boundary $\Gamma$ after the CMS-transformation.
Wave-Based Substructuring

The CMS-reduced model is used to obtain a subset of tuned blisk modes from which the modal wave deformation $\mathbf{W}$ along the blade-disk-interface is extracted. Following the Wave-Based Substructuring (WBS) by Donders (2008), an orthonormal basis is derived by a Singular Value Decomposition (SVD) of the interface waves. The consideration of waves associated to singular values above a specified limit yields a reduced modal basis $\mathbf{W}_{\text{svd}}$.

$$
\mathbf{u} = \left(\begin{array}{c}
u_{\Gamma} \\
\mathbf{q}_{\Xi}
\end{array}\right) = \mathbf{T}_{\text{wbs}} \left(\begin{array}{c}
u_{\Gamma} \\
\mathbf{q}_{\Xi}
\end{array}\right) \quad \text{where} \quad \mathbf{T}_{\text{wbs}} = \left[\begin{array}{cc}
\mathbf{W}_{\text{svd}} & 0 \\
0 & \mathbf{I}
\end{array}\right].
$$

(8)

An application of the transformation $\mathbf{T}_{\text{wbs}}$ reduces the amount of interface degrees of freedom.

Secondary Modal Truncation

To decrease the dimension of the reduced CMS/WBS-model, a subset of blisk modes $\Phi_{\text{smt}}$ of the mistuned system is used to form the transformation matrix $\mathbf{T}_{\text{smt}}$,

$$
\mathbf{u} = \mathbf{T}_{\text{smt}} \mathbf{q} \quad \text{where} \quad \mathbf{T}_{\text{smt}} = \Phi_{\text{smt}} \quad \text{and} \quad \mathbf{K}_{\text{cms/wbs}} \Phi_{\text{smt}} = \mathbf{M}_{\text{cms/wbs}} \Phi_{\text{smt}} \Lambda.
$$

(9)

This Secondary Modal Truncation (SMT) leads to the final set of reduced governing equations.

Comparison with the full model

To assess its accuracy and efficiency, the reduced order model is compared to the finite element representation of a blisk with ten bladed segments (see Figure 1). Fixed boundary conditions are applied along the inner surface of the disk’s central hole.

![Finite element models of a single segment and the full blisk (left). Comparison between the nodal diameter diagrams obtained from the full and reduced models (right).](image)

Figure 1: Finite element models of a single segment and the full blisk (left). Comparison between the nodal diameter diagrams obtained from the full and reduced models (right).

First, free blisk vibrations are evaluated in terms of a Modal Assurance Criterion (MAC) by Allemang and Brown (1982) and a relative eigenfrequency difference $\Delta f_{\text{rel},i}$ between the reduced and full finite element model,

$$
\text{MAC}_{i,j} = \frac{|\Phi_{\text{rom},i}^T \Psi_{\text{full},j}|^2}{\Phi_{\text{rom},i}^T \Phi_{\text{rom},i} \Psi_{\text{full},j}^T \Psi_{\text{full},j}} \quad \text{and} \quad \Delta f_{\text{rel},i} = \frac{f_{\text{rom},i} - f_{\text{full},i}}{f_{\text{full},i}}.
$$

(10)

The total amount of 8,610 dof of the FE model is reduced to 1,460 dof through retaining 1,260 dof along the blade-disk-interface, 10 component modes per blade and 10 mode families per harmonic index of the disk (i.e. 100 disk modes) inside the CMS-basis.
Despite a dimension reduction by 83%, the reduced order model sufficiently approximates the first 110 tuned blisk modes within a relative tolerance $\Delta f_{rel}{}$ of 1% and a MAC level above 0.96 (see Figure 2). An additional WBS-reduction with 100 waves along the blade-disk-interface hardly deteriorates the first 110 blisk modes. The first 100 blisk modes are used for an additional SMT-reduction in the following forced response analysis.

Next, the forced response of a mistuned blisk to an external EO2 forcing is analyzed. A random mistuning pattern is applied to the blisk in Figure 1 by scaling the stiffness of each blade $k$ by an individual mistuning factor $\kappa_k$ listed in Table 1. The vibrational response is monitored at the tip of each blade. The accuracy of the response prediction by ROMs with different reduction levels is assessed in Figure 3 by comparing the displacement amplitudes $\hat{u}^*$ and phase angles $\theta$ with the results of the unreduced FE model.

Figure 2: Modal assurance criterion (MAC$_{i,j}$) and relative frequency difference $\Delta f_{rel,i}$ between the reduced and full finite element model of a tuned blisk

Figure 3: Comparison of the forced EO2 response between the reduced and full finite element model of a mistuned blisk (left: ROM without SMT, right: ROM with SMT)
Table 1: Frequency mistuning factors of the blisk

<table>
<thead>
<tr>
<th>Blade $k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_k$</td>
<td>0.84</td>
<td>0.98</td>
<td>1.05</td>
<td>0.81</td>
<td>1.49</td>
<td>1.05</td>
<td>1.12</td>
<td>0.52</td>
<td>1.41</td>
<td>1.12</td>
</tr>
</tbody>
</table>

The excellent match of the models is confirmed quantitatively by the relative difference between the displacement amplitude $\hat{u}^{*}_{\text{ROM},k}$ of the ROM and the results $\hat{u}^{*}_{\text{FEA},k}$ of the FE simulation,

$$\Delta\hat{u}^{*}_{\text{rel},k} = \frac{|\hat{u}^{*}_{\text{ROM},k} - \hat{u}^{*}_{\text{FEA},k}|}{\hat{u}^{*}_{\text{FEA},k}} \quad \text{for} \quad k = 1, 2, \ldots, N .$$

Figure 4 shows that the maximum amplitude difference of the CMS-model and the CMS/SMT-model is below 0.3%. In case of a WBS-reduced interface, the amplitudes remain within a tolerance of 0.7% compared to the FE results.

The solution time of 29 s for the CMS/WBS/SMT-model yields a computational saving of 90% compared to the FE simulation time of 284 s. Since the reduction process requires 48 s, the overall computation time of the ROM (including the reduction and the forced response simulation) outperforms the FE simulation by a factor of almost four (saving 73% of time).

AERODYNAMIC COUPLING

The matrices $K_{\text{aero}}$ and $C_{\text{aero}}$ in Eq. (1) are made up of aerodynamic influence coefficients, which express the stiffening and damping influence between each blade and each mode. Their derivation is based on the linearized CFD simulation of unsteady pressure fluctuations. The flow perturbations are computed for a prescribed motion of a tuned blading in a specific mode shape with a particular inter-blade phase angle (IBPA) and frequency. In accordance with the fixed-interface CMS approach, the roots of the blades are kept fixed to ensure consistency between the aerodynamic coupling forces and the structural component modes.

Aerodynamic damping and stiffness

The aerodynamic coupling is derived as an equivalent viscous damping coefficient $c_v$,

$$c_v = \frac{W_{\text{cyc}}}{\pi \omega A^2} ,$$

where $W_{\text{cyc}}$ denotes the energy dissipated during a vibration cycle with amplitude $A$ and frequency $\omega$. To determine this cyclic aerodynamic work, the linearized RANS equations are
solved for the unsteady perturbation of the average flow. The unsteady displacement field \( x'_{i,r} \) of each structural mode \( i \) is then associated with a resulting local pressure at location \( r \), consisting of a stationary and an oscillating part (superscripts \( 0 \) and \( \prime \) respectively),

\[
p_i, r(x_r^0, x'_r, t) = p_i(r(x_r^0) + p_i, r(x'_r, x'_r, t)).
\] (13)

The pressure \( p_{i,r} \) yields the local force \( f_{i,r} \) at a surface element \( r \) of size \( S_r \) and orientation \( n_{i,r} \),

\[
f_{i,r} = p_{i,r} n_{i,r} S_r \quad \text{where} \quad n(x_r^0, x'_r, t) = n_i(r(x_r^0) + n'_i, r(x'_r, x'_r, t)).
\] (14)

The vector \( \mathbf{f}_i \) of all local forces is included in the following modal equation of motion:

\[
\Phi^H M \dot{\Phi} \dot{q} + \Phi^H K \Phi q = \mathbf{F}_i \quad \text{where} \quad u = \Phi q \quad \text{and} \quad \mathbf{F}_i = \Phi^H \mathbf{f}_i.
\] (15)

The modal matrix \( \Phi \) is formed column-wise by the mode shapes \( \varphi_j \). The right-hand side of the decoupled equation for mode \( j \) is expressed in terms of a damping coefficient \( c_{(j,i)} \),

\[
m_j \ddot{q}_j + k_j q_j = F_{j,i} \quad \Leftrightarrow \quad m_j \ddot{q}_j + k_j q_j = c_{(j,i)} \dot{q}_i.
\] (16)

Assuming a harmonic vibration of mode \( j \), the dissipated energy can be computed by integrating the right-hand side of Eq. (16) over one vibration period \( T \) (Kersken et al., 2012),

\[
W_{j,i} = \int_0^T -i\omega \int_0^T e^{-i\omega t} \varphi_j^H \mathbf{R} \{ \hat{\mathbf{f}} e^{i\omega t} \} \, dt = \pi \varphi_j^H \mathbf{f}_i.
\] (17)

The local aerodynamic work per area which is performed on the blade surface by the fluid is

\[
u_{(j,i)} = i\pi \varphi_j^H \mathbf{f}_i, r.
\] (18)

From the above equations it can be seen that the aerodynamic work is a complex quantity,

\[
\Re \{ W_{j,i} \} = \pi \left( \Re \{ \varphi_j^H \} \Re \{ \mathbf{f}_i \} + \Im \{ \varphi_j^H \} \Im \{ \mathbf{f}_i \} \right),
\] \( (19) \)

\[
\Im \{ W_{j,i} \} = -\pi \left( \Re \{ \varphi_j^H \} \Im \{ \mathbf{f}_i \} - \Im \{ \varphi_j^H \} \Re \{ \mathbf{f}_i \} \right).
\] (20)

By comparison with a damped single degree of freedom system in the frequency domain

\[
(-\omega^2 m_{\text{aero}} + i\omega c_{\text{aero}} + k_{\text{aero}}) \dot{q}_j = \hat{F}_{j,i} \quad \Leftrightarrow \quad \omega^2 m_{\text{aero}} + i\omega c_{\text{aero}} + k_{\text{aero}} = i\omega c_{(j,i)},
\] (21)

the equivalent damping coefficient \( c_{\text{aero}} \) splits up into an aerodynamic damping and stiffness:

\[
k_{\text{aero}}^{(j,i)} = \pi A^2 \quad \text{and} \quad c_{\text{aero}}^{(j,i)} = \pi A^2 \omega k_{\text{aero}}^{(j,i)}.
\] (22)

Due to the high blade-to-air mass ratio, the contribution of \( m_{\text{aero}} \) in Eq. (22) is omitted.

Equations (12) to (22) are valid for a single inter-blade phase angle. To implement the aerodynamic stiffness and damping in the ROM, they are transformed from the traveling wave mode representation into aerodynamic influence coefficients. Thus, the aerodynamic coupling has to be computed for all possible inter-blade phase angles and all relevant mode families.

In case of a forced response, the vibrations of bladed disks are composed of several modal components. Since aeroelasticity is treated linearly, the aerodynamic force of such a combined motion is covered by the influence coefficients of the participating modes. The interaction between modes \( j \) and \( i \) is computed by the transformation of the pressure fluctuation \( \mathbf{f}_j \) caused by a modal displacement \( j \) with the eigenvector \( \varphi_j \), mode \( i \) (see Eq. (15) and Eq. (16)),

\[
\mathbf{F}_{i,j} = \varphi_j^H \mathbf{f}_j.
\] (23)

For brevity, the aerodynamic stiffness and damping are synonymously denoted as a coupling quantity \( k_{\text{aero}}^{(i,j)} \) in the following sections.
Aeroelastic mode interaction

The presented approach implies linearity of the modeled system, which includes the structural dynamics as well as the aerodynamic coupling. To illustrate the consequences of this assumption on the aerodynamic mode interaction, an exemplary turbine cascade is analyzed. Unsteady CFD-simulations for three different blade mode shapes are performed: (1) a flapwise bending mode (1F), (2) a torsional mode (1T), and (3) a generic mode combination (1F/1T) of both mode shapes. Each mode shape is analyzed at the mean value of the 1F and 1T natural frequencies (see Figure 5).

By assuming linearity of the aeroelastic model, the generalized force for the bending-torsion-mode combination (1F/1T) is obtained from a superposition of the participating modal displacements \( \varphi \) and pressures \( f \),

\[
\varphi_{1F/1T}^H f_{1F/1T} = (\varphi_{1F}^H + \varphi_{1T}^H) (f_{1F} + f_{1T}) = \varphi_{1F}^H f_{1F} + \varphi_{1T}^H f_{1T} + \varphi_{1F}^H f_{1T} + \varphi_{1T}^H f_{1F}.
\]  

(24)

The comparison of the generalized force distributions obtained from the CFD analysis and from the superposition in Eq. (24) confirms the hypothesis of linearity and reveals the local significance of the aeroelastic inter-modal coupling terms \( \varphi_{1F}^H f_{1T} \) and \( \varphi_{1T}^H f_{1F} \) in Figure 6.

Figure 5: Computational flow domain for steady CFD simulation (left) and displacement contours of 1F mode, 1T mode, and combined 1F/1T mode for transient simulation (right)

Figure 6: Normalized profile distributions of the generalized aerodynamic force at blade midspan with and without the consideration of aerodynamic mode interaction (IBPA 180°)
Aerodynamic influence coefficients

The aerodynamic coupling \( k^{(i,j)}_{\text{aero},\sigma} \) for all inter-blade phase angles \( \sigma \) is transformed into influence coefficients \( k^{(i,j)}_{\text{aero}} \) by a complex Fourier transform (Crawley, 1988),

\[
K^{(i,j)}_{\text{aero}} = E \text{ diag} (k^{(i,j)}_{\text{aero},\sigma}) E^{-1}, \tag{25}
\]

where the Fourier matrix \( E \) is defined as

\[
E = \frac{1}{\sqrt{N}} \begin{bmatrix}
    e_{0,0} & e_{0,1} & \cdots & e_{0,N-1} \\
    e_{1,0} & e_{1,1} & \cdots & e_{1,N-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    e_{N-1,0} & e_{N-1,1} & \cdots & e_{N-1,N-1}
\end{bmatrix}, \quad e_{k,l} = e^{i \frac{2\pi}{N} kl} \quad \text{and} \quad k, l = 0, \ldots, N - 1. \tag{26}
\]

The cyclic coupling matrix \( K^{(i,j)}_{\text{aero}} \) comprises the aeroelastic inter-blade coupling between all blades \((k, l)\) for a single mode combination \((i, j)\). Each matrix element \( k^{(i,j)}_{\text{aero},k,l} \) represents the generalized aerodynamic force acting on mode \( i \) of blade \( k \) due to a vibrational motion of blade \( l \) in mode shape \( j \). Performing the transformation in Eq. (25) for all mode combinations yields the fully populated aerodynamic coupling matrix \( K_{\text{aero}} \).

APPLICATION OF THE REDUCED ORDER MODEL

The presented approach is used to analyze the dynamic behavior of an axial compressor blisk comprising 20 blades with NACA3506 profiles in Figure 7. The nodal diameter diagram confirms the accurate representation of the first five mode families by a ROM with 200 interface waves, 10 component modes per blade, and 10 modes per harmonic index of the disk (i.e. 200 disk modes). The frequencies of the 1F and 1T mode families at 189 Hz and 319 Hz, respectively, are well separated apart from the crossing of a disk-dominated mode family at nodal diameter ND3. The displacement contours in Figure 7 indicate that the enclosed ND3-blisk mode at 270 Hz features an interaction of the 1F and 1T modes besides the disk displacement.

![Finite element model of the compressor blisk (left), displacement contours of the ND3 blisk mode (center), and nodal diameter diagram of the ROM and FE model (right)](image)

Figure 7: Finite element model of the compressor blisk (left), displacement contours of the ND3 blisk mode (center), and nodal diameter diagram of the ROM and FE model (right)

Aerodynamic coupling of the axial compressor blisk

To reduce the amount of CFD simulations, the aerodynamic coupling \( k_{\text{aero}} \) is limited to the first five mode families, which are well approximated by the structural reduced order model. Since the following study focuses on the interaction of the first two mode families, this data is sufficient to cover the relevant aeroelastic effects. The results for the 1F and 1T modes are presented in terms of natural frequency \( f \) and damping ratio \( \zeta \) of the aeroelastic eigenvalue \( \lambda \),

\[
\zeta = \frac{\delta}{\omega} \quad \text{where} \quad \lambda = -\delta + i\omega. \tag{27}
\]
While the *intra-modal* coupling \((i, i)\) in Figure 8 provides positive damping for all inter-blade phase angles, the *inter-modal* coupling \((i, j)\) features negative damping values.

Figure 8: *Aerodynamic coupling between the 1F mode (index 1) and 1T mode (index 2)* (□: IBPA 0°, △: IBPA 90°, ♦: IBPA 180°, ▽: IBPA 270°, ○: various IBPA)

Figure 9 shows the magnitude of the associated influence coefficients \(\mid k_{aero}\mid\) for a section of 30 × 30 elements covering the interaction of three adjacent blades (10 modes per blade). The almost block diagonal dominance of the coefficients indicates that the aerodynamic coupling is governed by intra-blade forces. A detailed analysis of each 10 × 10 blade block along the main diagonal shows that beside the diagonal intra-modal elements, the inter-modal terms of specific mode combinations feature prominent influence levels. For instance, the aerodynamic forces of the second, third and fifth mode may have a significant influence on the first mode \((\mid k_{aero}^{(1,2)}\mid, \mid k_{aero}^{(1,3)}\mid, \mid k_{aero}^{(1,5)}\mid)\). The physical cause of this influence distribution is found in the displacement of the blade leading edge for modes 2, 3 and 5 (see Figure 9). Since periodic changes of the flow incidence result in fluctuations of the blades’ pressure distribution, the overall lifting force could easily induce a flapwise bending motion in mode shape 1.

Figure 9: *Magnitude of the aerodynamic influence coefficients for a section of three blades with 10 modes each (left) and displacement contours of the first five blade modes (right)*
Forced response and stability analysis

To reveal the effect of the aerodynamic coupling on the tuned and mistuned blisk dynamics, two different configurations are analyzed: (1) a blisk with a rigid disk and (2) a blisk with a flexible disk. The analyzed frequency range is focused on the eigenfrequencies of the 1F and 1T modes. The locations of force application and response monitoring are marked in Figure 7.

Compressor blisk with a rigid disk

To analyze the aerodynamic interaction apart from structural coupling effects, the stiffness of the disk is increased to provide an almost rigid component. The forced response for two different engine orders in Figure 10 illustrates the nodal-diameter dependence of the intra-modal aerodynamic coupling $k_{aero}^{(i,i)}$. The increase in aerodynamic 1T damping from IBPA 0° to 90° leads to a significant reduction of the second resonance amplitude from EO0 to EO5 (see Figure 8). Additionally, the consideration of inter-modal aerodynamic influences $k_{aero}^{(i,j)}$ barely affects the response due to the negligible mode interaction via the rigid disk.

![Figure 10: Forced response of the tuned blisk with a rigid disk for EO0 (left) and EO5 (right) without aerodynamic coupling ($k_{aero} = 0$), with intra-modal coupling $k_{aero}^{(i,i)}$, inter-modal coupling $k_{aero}^{(i,j)}$, and full coupling $k_{aero}$](image)

Next, the natural frequency of the 1F mode is artificially increased while decreasing the 1T eigenfrequency to obtain a generic combination of both mode families. The modal EO5 response in Figure 11 shows a strong interaction of the blade component modes $\phi_1$ (1F) and $\phi_2$ (1T). Since the inter-modal influence $k_{aero}^{(i,j)}$ of the dominant torsion mode provides negative damping at an inter-blade phase angle of 90° (see Figure 8), the response with consideration of the modal interaction $k_{aero}$ exceeds the maximum displacement amplitudes of the in-vacuo blisk ($k_{aero} = 0$) as well as the aeroleastic response without aerodynamic mode interaction ($k_{aero}^{(i,i)}$).

![Figure 11: Forced EO5 response of the tuned blisk with a rigid disk and 1F/1T interaction in the physical (left) and modal domain (right) without aerodynamic coupling ($k_{aero} = 0$), with intra-modal coupling $k_{aero}^{(i,i)}$, inter-modal coupling $k_{aero}^{(i,j)}$, and full coupling $k_{aero}$](image)
To address the issue of inter-modal coupling between adjacent blades, an alternating A/B-mistuning pattern is analyzed. By increasing the 1F eigenfrequency of blade type A and decreasing the 1T eigenfrequency of blade type B, an inter-blade correlation between both mode families is created (see Figure 12). In comparison to Figure 11, a smaller influence of the inter-modal coupling $k_{aero}^{(i,j)}$ on the forced EO5 response is observed. This stems from the rapid decay of the influence coefficients away from the main diagonal of the aerodynamic matrix (see Figure 9). In combination with the increase in aerodynamic damping by mistuning, the rigid disk prevents the localization of vibrational energy which attenuates the response amplitude.

![Figure 12: Forced EO5 response of the mistuned blisk with a rigid disk in the physical (left) and modal domain (right) without aerodynamic coupling ($k_{aero} = 0$), with intra-modal coupling $k_{aero}^{(i,i)}$, inter-modal coupling $k_{aero}^{(i,j)}$, and full coupling $k_{aero}$](image)

The beneficial effect of an alternating A/B-mistuning on the aeroelastic stability of the analyzed blisk is illustrated in Figure 13. First, the aeroelastic eigenvalues of the tuned system are computed. Since the central hole of the rigid disk is kept fixed, the disk’s outer annulus features a constraint displacement which resembles a fixed boundary condition for each of the attached blades. Consequently, the eigenvalues of this configuration are in good agreement with the unsteady CFD results presented in Figure 8 for a fixed-interface blade motion in mode 1F and in mode 1T. This comparison confirms the accurate integration of the aerodynamic blade coupling in the reduced order model.

![Figure 13: Comparison between the aeroelastic eigenvalues of the ROM and CFD simulations for the tuned and A/B-mistuned blisk (□: IBPA 0°, △: IBPA 90°, ◆: IBPA 180°, ▽: IBPA 270°, ○: various IBPA)](image)

On the contrary, an alternating A/B-mistuning of the blade stiffness yields two separate clusters of eigenvalues. It can be seen that both clusters feature a contraction in the range aerodynamic damping values $\zeta$. This indicates an increased stability margin of the least stable modes as well as a reduction of the maximum aerodynamic damping level.
Compressor blisk with a flexible disk

To study the effect of the aerodynamic coupling in regions of frequency veerings and crossings between closely spaced mode families, the forced EO3 response of the tuned compressor blisk is analyzed in Figure 14. While the resonance at 190 Hz is solely dominated by intra-modal 1F forces, the second and third resonances feature notable contributions of the 1F as well as the 1T component modes. Especially, the maximum displacement amplitude at 320 Hz shows an obvious effect of aerodynamic mode family interaction.

![Figure 14: Forced EO3 response of the tuned blisk with a flexible disk in physical (left) and modal domain (right) without aerodynamic coupling ($k_{\text{aero}} = 0$), with intra-modal coupling $k_{\text{aero}}^{(i,i)}$, inter-modal coupling $k_{\text{aero}}^{(i,j)}$, and full coupling $k_{\text{aero}}$.](image)

CONCLUSIONS

The extension of a substructure-based reduced order model to account for the aerodynamic coupling in mistuned bladed disks is presented. The aerodynamic interaction of the blading is included in the model by additional stiffness and damping matrices of complex influence coefficients. The inter-modal aerodynamic coupling terms are obtained from linearized CFD simulations by combining modal pressures and displacements of different mode families.

The reduction of a structural blisk model by 83% leads to a computational saving of 90% in comparison to unreduced finite element simulations. Relative frequency differences below 1% and amplitude deviations beneath 0.7% confirm the accurate representation of free and forced vibrations of tuned and mistuned blisk configurations. The prediction of inter-modal coupling forces by modal superposition of unsteady pressure fields is substantiated by unsteady CFD simulations. Finally, a comparison of the tuned aeroelastic eigenvalues with CFD results illustrates the accurate integration of the aerodynamic blade coupling in the structural reduced order model.

The practical applicability of the presented approach is demonstrated by means of a forced response analysis and an aeroelastic stability study of an axial compressor blisk. In this context, the nodal-diameter dependence of the intra-modal aerodynamic damping and the significance of the inter-modal aerodynamic coupling are highlighted. While the aerodynamic coupling between various mode families appears to be negligible for the hypothetic case of a tuned rigid disk, it may strongly affect the mistuned dynamics of flexible blisks in veering regions with closely spaced modes.

For future work, the presented approach will be extended to include frictional contact joints between shrouded blades. In view of the multi-harmonic content of nonlinear blade vibrations, the modeling of frequency-dependent aerodynamic coupling coefficients will be considered.
ACKNOWLEDGEMENTS

This work was supported by the German Research Foundation (DFG) and the Research Association for Combustion Engines (FVV) eV within the framework of the co-funded project “Mistuning of bladed disks with aerodynamic and structural coupling”. Additional support was provided by the DFG through the Collaborative Research Center (SFB) 871, Regeneration of Complex Capital Goods.

REFERENCES


